Gibbs Conditioning and Large Deviation in Portfolio Risk

Portfolio LDP

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Motivation and backgrounds

Portfolio risk

- A portfolio contains n obligors. The obligor defaults when the value of the asset falls below a given threshold.
- Value of the asset is commonly estimated by a factor model:

$$Y_i = \sum_{j=1}^{k} a_j W_j + b\epsilon_i, \quad i = 1, 2, \ldots, n,$$

where $\sum_{j=1}^{k} a_j^2 + b^2 = 1$.

- Y_i is the i^{th} asset value. $\{W_j : j = 1, \ldots, k\}$ are common factors shared between obligors. $\{\epsilon_i : i = 1, \ldots, n\}$ are idiosyncratic factors and are independent and identically distributed (i.i.d.) and independent of $\{W_j\}$.

Portfolio risk

- We focus on the extreme loss of the portfolio.
- The total loss L_n is defined to be

$$L_n = \sum_{i=1}^{n} U_i X_i,$$

where $X_i = \mathbb{1}_{[Y_i \le v]}$. v is the default threshold.

- The random variables $\{U_i\}$ are i.i.d. and interpreted as loss given default (LGD) and independent of $\{W_j\}, \{\epsilon_i\}$.

Motivation

- Let $Z_k = \sum_{j=1}^{k} a_j W_j$ represent the effect of common factors.
- If $\boldsymbol{E}[U_1] < \infty$, then as $n \to \infty$, $\frac{L_n}{n} \to p(Z_k) = \boldsymbol{E}[U_1] F_\epsilon\left(\frac{v - Z_k}{b}\right) \in [0, \boldsymbol{E}[U_1]]$.
- The conditional central limit theorem:

$$\lim_{n\to\infty} \boldsymbol{P}\left(\sqrt{n}\left(\frac{L_n}{n} - p(Z_k)\right) \leq x \middle| Z_k\right) = \boldsymbol{P}\left(N(0, \sigma^2(Z_k)) \leq x | Z\right),$$

where $\sigma^2(Z_k) = \boldsymbol{Var}[U_1 X_1 | Z_k]$.

- Glasserman et al. (2007), Collamore et al. (2022), and Bassamboo et al. (2008) have studied logarithmic and sharp tail probability estimates associated with $\boldsymbol{P}(L_n \geq n x_n)$ for a suitable $\{x_n\}$ increasing to $\boldsymbol{E}[U_1]$.

Motivation (Contd.)

Motivated by this, we investigate the following:

- We allow the number of common factors diverge and assume $Z_{k_n} \to Z_\infty$ in probability.
    - Z_∞ is degenerate at a constant.
    - Z_∞ is a non-degenerate random variable.

- We study the sharp tail behavior of $\boldsymbol{P}\left(\frac{L_n}{n} > x\right)$, where $x \in (\boldsymbol{E}[U_1], \infty)$.

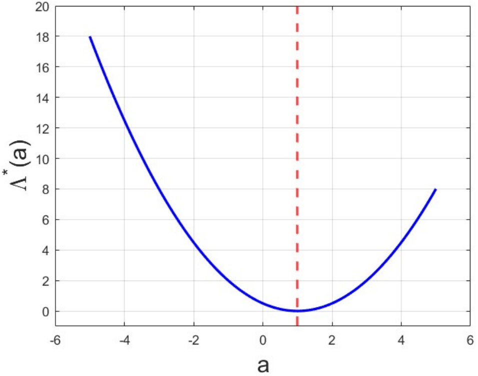
Large deviation theorem

- **Recall the large deviation problem in the i.i.d. case**: Let $X_1, \cdots, X_n$ be i.i.d. with mean μ, and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Assume that $\boldsymbol{E}[e^{\theta X_1}] < \infty$ for $\theta \in (-\delta, \delta)$ for some $\delta > 0$.
  - Cramer's Theorem: $\lim\limits_{n\to\infty} \frac{1}{n} \log \boldsymbol{P}(\bar{X}_n > a) = -\Lambda^*(a)$.
  - Bahadur-Rao Theorem: $\lim\limits_{n\to\infty} \sqrt{n} e^{n\Lambda^*(a)} \boldsymbol{P}(\bar{X}_n > a) = C_a$.

  Rate function: $\Lambda^*(a) = \sup_\theta\{\theta a - \log \boldsymbol{E}[e^{\theta X_1}]\}$, $a > \mu$.
- $\Lambda^*(a)$ is non-negative, convex, lower-semi continuous and $\Lambda^*(\mu) = 0$.

Large deviation theorem

- **Example**: Let $X_1, \cdots, X_n$ be i.i.d. with $N(1,1)$. Then $\Lambda^*(a) = \frac{(a-1)^2}{2}$.



$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\left|\bar{X}_n - 1\right| > a\right) = -\frac{a^2}{2}$$

Portfolio risk

Large deviation results

Intuition

- Notice that $\frac{L_n}{n} \to p(Z_\infty) = \boldsymbol{E}[U_1]F_\epsilon\left(\frac{v-Z_\infty}{b}\right)$ and $p(Z_\infty) \in [0, \boldsymbol{E}[U_1]]$.
- For $x > \boldsymbol{E}[U_1]$, we study the behavior of $\frac{L_n}{n}$ beyond its limiting support.
- When $\frac{L_n}{n} > x$, the following events occur simultaneously:
  - All obligors default, which implies $Z_\infty \to -\infty$.
  - The loss given default for all obligors is large; specifically, $\boldsymbol{E}[U_1]$ is shifted to x.

Large deviation theorem

- The large deviation probability $\boldsymbol{P}\left(\frac{L_n}{n} > x\right)$ is highly influenced by the tail properties of Z_∞ and ϵ.
- Our analysis of $\boldsymbol{P}\left(\frac{L_n}{n} > x\right)$ considers cases where Z_∞ and ϵ are Gaussian, Laplace, or regularly varying distributions.

Large deviation theorem (Contd.)

Under regularity conditions, such as $\Lambda_U'(\theta) = x$ has unique root θ_x in $(0, \theta_0)$, we derive the following theorem:

Large deviation theorem

There exists a sequence of real-valued function $\phi_n(\cdot)$, a sequence of constants M_n, and a constant K_x, such that

$$\lim_{n\to\infty} \sqrt{n} e^{n\Lambda_U^*(x)} \exp(n\phi_n(-M_n)) \boldsymbol{P}(L_n > nx) = K_x.$$

$\Lambda_U^*(\cdot)$ is the rate function of U. K_x depends on x, v, variance of Z_∞ and ϵ.

Large deviation theorem (Contd.)

Idea of the theorem:

$$\boldsymbol{P}(L_n > nx) = \int_{-\infty}^{\infty} \boldsymbol{P}\left(L_n > nx | Z_{k_n} = z\right) \mathrm{d}F_{Z_{k_n}}(z)$$
$$\sim C_x n^{-\frac{1}{2}} e^{-n\Lambda_U^*(x)} \int_{-\infty}^{\infty} e^{-n\phi_n(z)} f_{Z_\infty}(z) \mathrm{d}z.$$

- One can **not** apply standard Laplace method to the integral $\int_{-\infty}^{\infty} e^{-n\phi_n(z)} f_{Z_\infty}(z)\mathrm{d}z$, since the critical point is $z = -M_n \to -\infty$ as $n \to \infty$.
- We decompose the integral into three parts: $\int_{-\infty}^{-M_n(1+\beta)}$, $\int_{-M_n(1+\beta)}^{-M_n(1-\beta)}$, and $\int_{-M_n(1-\beta)}^{\infty}$, for some constant $\beta > 0$.

Large deviation theorem (Contd.)

Choice of M_n depends on the distribution assumption of Z_∞ and ϵ.

- If $\epsilon \sim N(0, \sigma_\epsilon^2)$ and $Z_\infty \sim N(0, \sigma_{Z_\infty}^2)$ then $M_n \sim \sqrt{\log n}$ and

$$\exp(n\phi_n(-M_n)) = n^{\frac{b^2\sigma_\epsilon^2}{\sigma_{Z_\infty}^2}} [e^{\sqrt{\log n}}]^{-\frac{\sqrt{2}vb\sigma_\epsilon}{\sigma_{Z_\infty}^2}} (\log n)^{\frac{1}{2}-\frac{b^2\sigma_\epsilon^2}{2\sigma_{Z_\infty}^2}}.$$

- If $\epsilon \sim Lap(0, \sigma_\epsilon)$ and $Z_\infty \sim Lap(0, \sigma_{Z_\infty})$ then $M_n \sim \log n$ and

$$\exp(n\phi_n(-M_n)) = n^{\frac{b\sigma_{Z_\infty}}{\sigma_\epsilon}}.$$

Large deviation theorem (Contd.)

Denote by $X \sim RV_\alpha$ a random variable with symmetric regularly varying tails, whose density satisfies $f_X(z) \sim \alpha_X |z|^{-\alpha_X - 1} L_X(z)$ as $|z| \to \infty$, where $L_X(\cdot)$ is a slowly varying function.

- If $\epsilon \sim RV_{\alpha_\epsilon}$ with slowly changing function $L_\epsilon(\cdot)$ and $Z_\infty \sim RV_{\alpha_{Z_\infty}}$ with slowly changing function $L_{Z_\infty}(\cdot)$, then $M_n \sim n^{\frac{1}{\alpha_\epsilon}}$ and

$$\exp(n\phi_n(-M_n)) = n^{\frac{\alpha_{Z_\infty}}{\alpha_\epsilon}} [L_\epsilon(n^{\frac{1}{\alpha_\epsilon}})]^{\frac{\alpha_{Z_\infty}}{\alpha_\epsilon}} [L_{Z_\infty}(n^{\frac{1}{\alpha_\epsilon}})]^{-1}.$$

- If $\epsilon \sim N(0, \sigma_\epsilon^2)$ and $Z_\infty \sim RV_{\alpha_{Z_\infty}}$ with slowly changing function $L_{Z_\infty}(\cdot)$ then $M_n \sim \sqrt{\log n}$ and

$$\exp(n\phi_n(-M_n)) = (\log n)^{\frac{(\alpha_{Z_\infty}+1)}{2}} \left[L_{Z_\infty}\left(\sqrt{\log n}\right)\right]^{-1}.$$

Gibbs conditioning principle

Gibbs conditioning principle

Under the same conditions as in the large deviation theorem, for any Borel set B

$$\lim_{n\to\infty} \boldsymbol{P}\left(U_1^{(n)} \in B, X_1^{(n)} = 1 | L_n > nx\right) = \boldsymbol{P}_{\theta_x}(B),$$

where

$$\boldsymbol{P}_{\theta_x}(B) = (\lambda_U(\theta_x))^{-1} \int_B e^{\theta_x y} dP_U(y),$$

and $P_U(\cdot)$ is the distribution of U and $\lambda_U(\theta) = \boldsymbol{E}[e^{\theta U}]$.

This implies that when $\frac{L_n}{n} > x$, all obligors default and the mean of U_1 (loss given default) shifts from $\boldsymbol{E}[U_1]$ to x.

Value at risk and expected shortfall

The large deviation theorem can be used to estimate the Value-at-Risk (VaR) and Expected Shortfall (ES) of a portfolio.

VaR

Value-at-Risk at level α, denoted by $x_{\alpha,n}$, is approximately estimated by

$$x_{\alpha,n} \approx \mathbf{E}[U] + \left[\frac{2\boldsymbol{Var}[U](-\log(1-\alpha) - \phi_n(-M_n) + \log K_x)}{n}\right]^{\frac{1}{2}}.$$

This approximation holds in the regime of extreme losses; that is, when α is close to 1.

Value at risk and expected shortfall (Contd.)

Expected shortfall is the expected loss conditional on the loss exceeding the Value-at-Risk threshold $x_{\alpha,n}$.

ES

The related expected shortfall, denoted by $\mathrm{ES}_\alpha\left(\frac{L_n}{n}\right)$, is approximately given by

$$\mathrm{ES}_\alpha\left(\frac{L_n}{n}\right) \approx x_{\alpha,n} + \frac{K_x e^{-n\phi_n(-M_n)}}{1-\alpha} \cdot \frac{e^{-n\Lambda_U^*(x_{\alpha,n})}}{n(\Lambda_U^*)'(x_{\alpha,n})}.$$

Portfolio risk

Concluding Remarks

Concluding Remarks

- We derived sharp large deviation estimates for the total loss L_n beyond the limiting support of $\frac{L_n}{n}$.
- We presented the Gibbs conditioning principle and illustrated the conditions under which extreme loss events occur.
- We provided approximations for Value-at-Risk (VaR) and Expected Shortfall (ES) in the extreme loss regime.

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