Sharp Large Deviation in Branching Process with Immigration

BPI LDP

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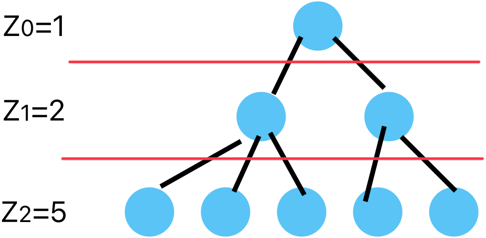
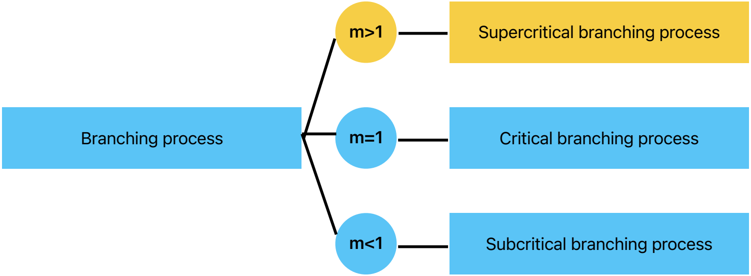


- Each individual reproduces a random number of children, following the same probability distribution.

introduction to BP

Branching processes are classified into three types: **subcritical**, **critical**, and **supercritical** cases.

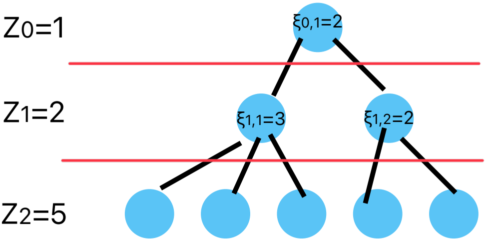
- $\boldsymbol{P}(Z_1 = j) = p_j$, and $\boldsymbol{E}(Z_1) = m$.



Large deviation problem in BP

Let $\{\xi_{n,i} : n \geq 0, i \geq 1\}$ represent i.i.d. random variables with the offspring distribution $\{p_j : j \geq 0\}$. We analyze the following ratio:

$$R_n = \frac{Z_{n+1}}{Z_n} = \frac{1}{Z_n}\sum_{i=1}^{Z_n}\xi_{n,i}, \quad i.e. \quad R_1 = \frac{1}{Z_1}\sum_{i=1}^{Z_1}\xi_{n,i}.$$



Large deviation problem in BP (Contd.)

- $R_n = \bar{\xi}_{Z_n}$ behaves like an estimator of the offspring mean, m.
- Athreya and Vidyashankar (1993) showed that for $a > m$ and $\boldsymbol{E}[e^{\theta \xi_{1,1}}] < \infty$ for $\theta \in (-\delta, \delta)$:

$$\lim_{n \to \infty} \gamma^{-n} \boldsymbol{P}(R_n > a | Z_n > 0) = C_a < \infty.$$

  where $\gamma = f'(q) > 1$, q is the extinction probability and $f(\cdot)$ is the probability generating function, $f(s) = \boldsymbol{E}[s^{Z_1}]$. This implies

$$\lim_{n \to \infty} \frac{1}{n} \log \boldsymbol{P}(R_n > a | Z_n > 0) = -\log \gamma.$$

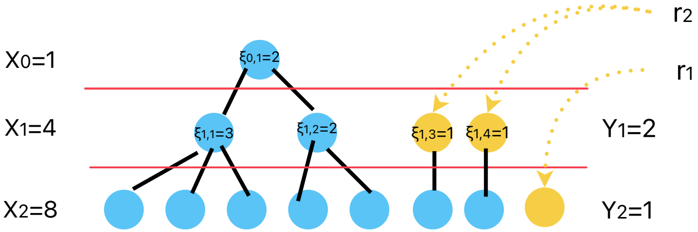
- The Cramer rate function is constant here since the main contribution comes from small values of Z_n.

Motivation

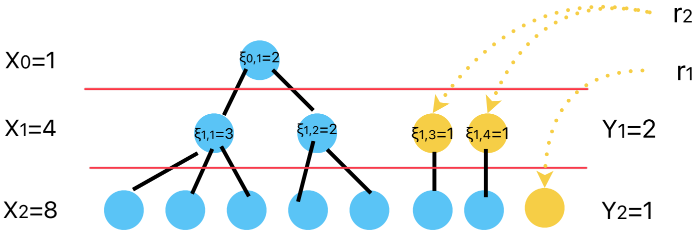
- To bring out the rate function, impose the condition $Z_n \geq v_n$, where $v_n \to \infty$.
  - Ney and Vidyashankar (2004) analyzed the behavior of $\boldsymbol{P}(R_n > a | Z_n \geq v_n)$ under various choices of v_n.
- Without conditioning on $X_n \geq v_n$, the rate function is hidden in branching processes with immigration (BPI) (Li and Li (2021)). We extend the above idea to BPI.
- We study 2 cases for v_n:
  - $v_n = o(m^n)$
  - $v_n = O(m^n)$

introduction to BPI

- In each generation, j immigrants arrive with probability r_j.
- Immigrants have the same offspring distribution.



Introduction to BPI (Contd.)

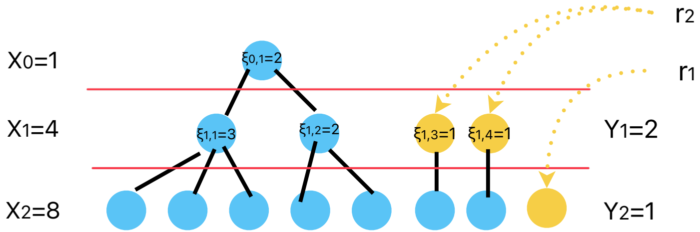


- X_i is the population size of the i^{th} generation. Y_i is the number of immigrants in i^{th} generation.
- $X_{n+1} = \sum_{j=1}^{X_n} \xi_{n,j} + Y_{n+1}$.

Introduction to BPI (Contd.)

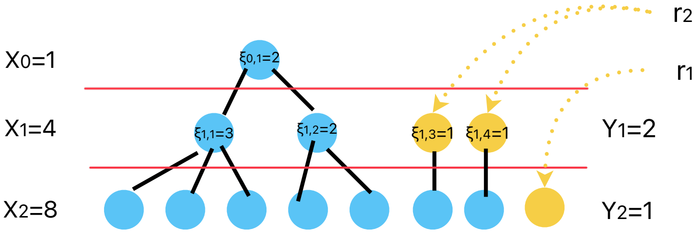
- Probability generating functions:
  - Immigration distribution: $g(s) = \boldsymbol{E}[s^{Y_1}] = \sum_{j \geq 0} s^j \cdot r_j$.
  - Offspring distribution: $h(s) = \boldsymbol{E}[s^{\xi_1}] = \sum_{j \geq 0} s^j \cdot p_j$.
  - For X_n: $f_n(s) = \boldsymbol{E}[s^{X_n}] = h_n(s) \cdot \prod_{i=0}^{n-1} g(h_i(s))$.
- $q \in [0, 1]$ is the extinction probability of BP without immigration, which satisfies $h(q) = q$.
- $m = h'(1)$ represents the offspring mean.

Introduction to BPI (Contd.)



- $q = 0$, $g(q) = 0$: Blue items never become extinct. At least 1 item immigrates in each generation. $X_n \geq n + 1$.
- $q = 0$, $g(q) > 0$: Blue items never become extinct. Some generations may have zero immigrants.

Introduction to BPI (Contd.)



- $q > 0$: Blue items may become extinct. Yellow items make more contribution to the population.

Introduction to BPI (Contd.)

- Under assumption $\boldsymbol{E}[\xi_{1,1} \log_+ \xi_{1,1}] < \infty$ and $\boldsymbol{E}[\log_+ Y_1] < \infty$,

$$\lim_{n\to\infty} \frac{X_n}{m^n} = V,$$

where V is a non-degenerate random variable. See Seneta (1970) for details.

- Our large deviation results are built under these assumptions, and $\boldsymbol{E}[e^{\theta \xi_{1,1}}] < \infty$ and $\boldsymbol{E}[e^{\theta Y_1}] < \infty$ for $\theta \in (-\delta, \delta)$.

- We focus on the case $m > 1$ and $p_0 + p_1 \in (0, 1)$, commonly referred to as the Schröder case.

Branching processes with immigration

Large deviation and local limit theorems

Limiting probability generating functions

- Limiting function for BP:

$$Q(s) = \lim_{n\to\infty} \frac{h_n(s) - q}{[h'(q)]^n}.$$

- Limiting function for case $q > 0$ and $g(q) > 0$:

$$Q^{(1)}(s) = \lim_{n\to\infty} \frac{f_n(s)}{[g(q)]^n} = \sum_{j=0}^{\infty} q_j^{(1)} \cdot s^j.$$

Limiting probability generating functions (Contd.)

- Limiting function for case $q = 0$, $g(q) > 0$, $p_1 > 0$, and $r_0 > 0$:

$$Q^{(2)}(s) = \lim_{n\to\infty} \frac{f_n(s)}{[p_1 r_0]^n} = \sum_{j=0}^{\infty} q_j^{(2)} \cdot s^j.$$

- Limiting function for case $q = 0$, $g(q) = 0$ $(r_0 = 0)$, $p_1 > 0$, $r_1 > 0$:

$$Q^*(s) = \lim_{n\to\infty} \frac{f_n(s)}{\left(r_1 \cdot p_1^{\frac{1}{2}} Q(s)\right)^n \cdot p_1^{\frac{n^2}{2}}}.$$

- Turning to the large deviation problem, assume $\boldsymbol{E}[e^{\theta \xi_{1,1}}] < \infty$ and $\boldsymbol{E}[e^{\theta Y_1}] < \infty$ for $\theta \in (-\delta, \delta)$. The rate function is defined as

$$\Lambda^*(a) = \sup_{\theta}\{\theta a - \log \boldsymbol{E}[e^{\theta \xi_{1,1}}]\}.$$

Unconditional large deviation

- $q = 0$, $g(q) = 0$, in this case $X_n \geq n + 1$.

$$\lim_{n\to\infty} \frac{\sqrt{n}}{p_1^{\frac{n^2+n}{2}} \cdot r_1^n \cdot \left[Q(e^{-\Lambda^*(a)})\right]^n} \cdot \boldsymbol{P}\left(R_n > a\right) = C_a \cdot Q^*(e^{-\Lambda^*(a)}),$$

  where C_a is the Bahadur constant of the offspring distribution, $\Lambda^*(\cdot)$ is the rate function of the offspring distribution.

- **Intuitive idea**: The primary contribution to $\boldsymbol{P}(R_n > a)$ comes from the values of X_n close to $n + 1$.

Unconditional large deviation (Contd.)

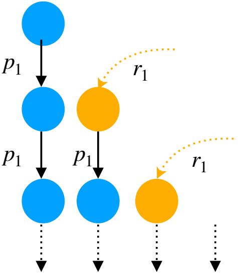
- Intuition for $\boldsymbol{P}(X_n = n+1)$

$X_0 = 1$

$X_1 = 2$

$X_2 = 3$

$X_n = n + 1$



$$\mathbf{P}(X_0 = 1) = 1$$

$$\mathbf{P}(X_1 = 2) = \mathbf{P}(X_1 = 2 \mid X_0 = 1)\mathbf{P}(X_0 = 1) = p_1 r_1$$

$$\mathbf{P}(X_2 = 3) = \mathbf{P}(X_2 = 3 \mid X_1 = 2)\mathbf{P}(X_1 = 2) = P_1^2 r_1 (p_1 r_1) = p_1^{1+2} r_1^2$$

$$\mathbf{P}(X_n = n + 1) = \mathbf{P}(X_n = n + 1 \mid X_{n-1} = n)\mathbf{P}(X_{n-1} = n) = p_1^{1+2+\cdots+n} r_1^n = p_1^{\frac{n^2+n}{2}} r_1^n$$

- Other terms come from the large deviation calculation.

Conditional large deviation

Conditional large deviation

For other cases, we study the conditional large deviation:

$$\boldsymbol{P}(R_n > a | X_n \geq v_n).$$

- $v_n = O(m^n)$, as $n \to \infty$, $\frac{v_n}{m^n} \to c > 0$.
- $v_n = o(m^n)$, as $n \to \infty$, $v_n \to \infty$.

Local limit theorem

Local limit theorem

If $v_n = O(m^n)$, and as $n \to \infty$, $\frac{v_n}{m^n} \to c > 0$, then

$$\left| m^n \cdot \mathbf{P}(X_n = v_n) - w_V\left(\frac{v_n}{m^n}\right) \right| \leq C \left(\frac{v_n}{m^n}\right)^{-1} \cdot \eta_0^n,$$

where $\eta_0 \in (0,1)$. η_0 depends on the offspring distribution and the immigration distribution. $w_V(\cdot)$ is the density function of V.

This implies $\lim_{n\to\infty} m^n \cdot \mathbf{P}(X_n = v_n) = w_V(c)$.

Local limit theorem (Contd.)

- Turn to the case $v_n = o(m^n)$.
- Let $\tilde{k}_n = n - \frac{\log v_n}{\log m}$ and $c_n = m^{\tilde{k}_n - [\tilde{k}_n]}$. Notice that c_n may oscillate between 1 and m.
- Define a sequence of random variable $\{T_j : j \geq 1\}$ as $T_j = V + \sum_{k=1}^{j-1} W^{(k)}$, where $\{W^{(k)}\}_{k\geq 1}$ are i.i.d. with distribution of W. W is the limiting distribution of $\frac{Z_n}{m^n}$ in BP. Denote the density function of T_j by $w_{T_j}(\cdot)$.
- Define $L_1(x) = \frac{1}{x} \sum_{j\geq 0} q_j^{(1)} w_{T_j}(\frac{1}{x})$ and $L_2(x) = \frac{1}{x} \sum_{j\geq 0} q_j^{(2)} w_{T_j}(\frac{1}{x})$

Local limit theorem (Contd.)

Local limit theorem

If $v_n = o(m^n)$, $q > 0$ ($g(q) > 0$), and $\alpha = -\frac{\log g(q)}{\log m} \in (0,1)$, then

$$\lim_{n\to\infty} \frac{\mathbf{P}(X_n = v_n)}{[g(q)]^n v_n^{\alpha-1} c_n L_1(c_n)} = 1.$$

Local limit theorem

If $v_n = o(m^n)$, $q = 0$, $g(q) > 0$ ($r_0 > 0$), $p_1 > 0$ and $\alpha = -\frac{\log(p_1 r_0)}{\log m} \in (0,1)$, then

$$\lim_{n\to\infty} \frac{\mathbf{P}(X_n = v_n)}{[p_1 r_0]^n v_n^{\alpha-1} c_n L_2(c_n)} = 1.$$

Conditional large deviation

- Similar to large deviations in BP, the primary contribution to $\boldsymbol{P}(R_n > a|X_n \geq v_n)$ comes from values of X_n close to v_n.
- Different rates of $\boldsymbol{P}(R_n > a|X_n \geq v_n)$ come from different local limit behavior of $\boldsymbol{P}(X_n = v_n)$ for $v_n = o(m^n)$, $v_n = O(m^n)$.
- Next, we turn to large deviation theorems of $\boldsymbol{P}(R_n > a|X_n \geq v_n)$ under conditions $\boldsymbol{E}[e^{\theta \xi_{1,1}}] < \infty$ and $\boldsymbol{E}[e^{\theta Y_1}] < \infty$ for $\theta \in (-\delta, \delta)$.

Conditional large deviation (Contd.)

Large deviation theorem

If $v_n = O(m^n)$, and as $n \to \infty$, $\frac{v_n}{m^n} \to c > 0$, then

$$\lim_{n\to\infty} \sqrt{v_n} \cdot e^{v_n \cdot \Lambda^*(a)} \cdot m^n \cdot \mathbf{P}\left(\frac{X_{n+1}}{X_n} > a \middle| X_n \geq v_n\right) = C_{a,c},$$

where $C_{a,c} = \frac{C_a \cdot w_V(c)}{\mathbf{P}(V>c)\cdot\left(1-e^{-\Lambda^*(a)}\right)}$. C_a is the Bahadur constant.

Conditional large deviation (Contd.)

Large deviation theorem

If $v_n = o(m^n)$, $q > 0$ ($g(q) > 0$), and $\alpha = -\frac{\log g(q)}{\log m} \in (0,1)$, then

$$\lim_{n\to\infty} \frac{\sqrt{v_n} \cdot e^{v_n \cdot \Lambda^*(a)}}{[g(q)]^n v_n^{\alpha-1} c_n L_1(c_n)} \cdot \mathbf{P}\left(\frac{X_{n+1}}{X_n} > a | X_n \geq v_n\right) = \frac{C_a}{\left(1 - e^{-\Lambda^*(a)}\right)}.$$

Large deviation theorem

If $v_n = o(m^n)$, $q = 0$, $g(q) > 0$ ($r_0 > 0$), $p_1 > 0$, and $\alpha = -\frac{\log(p_1 r_0)}{\log m} \in (0,1)$, then

$$\lim_{n\to\infty} \frac{\sqrt{v_n} \cdot e^{v_n \cdot \Lambda^*(a)}}{[p_1 r_0]^n v_n^{\alpha-1} c_n L_2(c_n)} \cdot \mathbf{P}\left(\frac{X_{n+1}}{X_n} > a | X_n \geq v_n\right) = \frac{C_a}{\left(1 - e^{-\Lambda^*(a)}\right)}.$$

Branching processes with immigration

Concluding Remarks

Concluding Remarks

- We derived local limit theorems for BPI under $v_n = o(m^n)$ and $v_n = O(m^n)$.
- We established conditional Large deviation theorem for R_n under different conditions of v_n.

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